# New Integral Inequalities of Hadamard's for Harmonically Convex Stochastic Processes on n-coordinates

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**ABSTACT.** In this study, we consider harmonically convex stochastic processes on n-coordinates. Also, we obtain Hermite-Hadamard type integral inequalities for these processes on n-coordinates.

**Keywords:** Harmonically convexity, stochastic process on n-coordinates, mean-square integral, Hermite-Hadamard inequality.

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## 1. INTRODUCTION

It is well known that, for every real convex function $f$ on the interval $[a,b]$, we have [1]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

These are celebrated Hermite-Hadamard inequalities. In probabilistic words, they say that

$$f(EX) \leq_{cx} Ef(X) \leq_{cx} Ef(X^*), f \in C_{cx}$$

where $E$ denotes mathematical expectation, $X$ (respectively, $X^*$) is a random variable having the uniform distribution on the interval $[a,b]$ (respectively, on the set $\{a,b\}$), $C_{cx}$ is the set of all real convex functions on $[a,b]$ and $\leq_{cx}$ stands for the so called convex order of random variables [2].

Alomari [3] generalized the classical Hermite-Hadamard type inequalities for every convex function and for every positive convex function on $[a,b]$. Mwaeze [4] obtained some generalized Hermite-Hadamard's inequality for every convex function and for every positive convex function on the coordinates. Elahi [5] obtained Hadamard's inequality for s-convex function on n-coordinates.

There are many studies in recent years on some types of convexity for stochastic processes and Hermite-Hadamard inequalities for related convex stochastic processes, and it is of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities in the literature. Convex stochastic processes were proposed and some properties were given for classical convex stochastic processes by Nikodem [6]. Stochastic convexity and its applications were defined by Shaked et al [7]. Jensen-convex, $\lambda$-convex stochastic processes were introduced by Skowronski [8]. The classical Hermite-Hadamard inequality to convex stochastic processes was extended by

Kotrys [9]. Convex stochastic processes on the coordinates were introduced and Hermite-Hadamard type inequalities for these processes were obtained by Set et al [10]. Harmonically convex stochastic processes were defined and Hermite-Hadamard type inequalities were obtained by Okur et al [11]. Okur et al [12] extended these processes on two-dimensional and interval obtained Hermite-Hadamard type inequalities and estimation for these processes. Karahan et al [13] investigated the convex stochastic processes on n-coordinates, and obtained Hermite-Hadamard type inequality for these processes.

The authors' findings led to our motivation to build our work. The main subject of this paper is to adapt some obtained results concerning harmonically convex functions on n-coordinates by Viloria et al [14] to harmonically convex stochastic processes on n-coordinates and to obtain Hermite-Hadamard type inequalities.

## 2. PRELIMINARIES

There are well-known definitions of stochastic processes and some fundamentals about Hermite Hadamard Inequality for stochastic processes in the literature (see, [9]-[13]).

**Definition 2.1[9].** *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $I \in \Im$, $I \subset \mathbb{R}$ be an interval. The stochastic process $X: I \times \Omega \to \mathbb{R}$ is called almost everywhere convex if*

$$X(\lambda t + (1-\lambda)s, \cdot) \leq \lambda X(t, \cdot) + (1-\lambda) X(s, \cdot)$$

*for all $t, s \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed then $X$ is said to be concave.*

**Definition 2.2[9].** *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $I \in \Im$, $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is called*

***(i)*** *continuous in probability on $I$ if for all $t_0 \in I$ if*

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

*where $P-\lim$ denotes limit in probability,*

***(ii)*** *mean-square continuous on $I$ if for all $t_0 \in I$ if*

$$\lim_{t \to t_0} E[X(t, \cdot) - X(t_0, \cdot)]^2 = 0,$$

*where $E[\xi(t, \cdot)]$ denotes expectation value of the random variable $X(t, \cdot)$,*

***(iii)*** *mean-square differentiable at a point if $t \in I$ if there is a random variable $X'(t, \cdot): I \times \Omega \to \mathbb{R}$ such that*

$$X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval $I$.

**Definition 2.3[9].** *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $I \in \Im$, $I \subset \mathbb{R}$ be an interval and $X: I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t, \cdot)^2] \leq \infty$ for all $t \in I$. Let*

*$[0,t] \subset I,\ 0 = t_0 < t_1 \dots t_n = t$ be a partition of $[0,t]$ and $\Theta_k \in [t_{k-1}, t_k]$ arbitrary for $k = 1, \dots, n$. A random variable $\eta: \Omega \times \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[0,t]$ if the following identity holds:*

$$\lim_{n\to\infty} E\left(\sum_{k=1}^{n} X(\Theta_k,\cdot)\,.\,(t_k - t_{k-1}) - \eta(t,\cdot)\right)^2 = 0.$$

*Then we can write almost everywhere*

$$\int_0^t X\,(u,\cdot)du = \eta(t,\cdot).$$

*The mean-square integral operator is increasing on $[0,t]$ almost everywhere, that is,*

$$X(t,\cdot) \le Y(t,\cdot) \ \Rightarrow \ \int_0^t X\,(t,\cdot)dt \le \int_0^t Y(t,\cdot)dt.$$

Let us consider the Hermite-Hadamard integral inequality for harmonically convex stochastic processes:

**Definition 2.4[11].** *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A stochastic process $X: I \times \Omega \to \mathbb{R}$ is said to be a harmonically convex stochastic process almost everywhere, if*

$$X\left(\frac{ts}{\lambda t + (1-\lambda)s},\cdot\right) \le \lambda X(s,\cdot) + (1-\lambda)X(t,\cdot) \tag{2.1}$$

*for all $t, s \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then $X$ is said to be a harmonically concave almost everywhere.*

**Proposition 2.1[11].** *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $X: I \times \Omega \to \mathbb{R}$ is a stochastic process.*

*(i) If $X$ is convex and nondecreasing stochastic process, then $X$ is harmonically convex.*

*(ii) If $X$ is harmonically convex and nonincreasing stochastic process, then $X$ is convex.*

**Theorem 2.3[11].** *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $X: I \times \Omega \to \mathbb{R}$ be a harmonically convex stochastic process and $a, b \in I^{\circ}$ with $a < b$. If $X \in L[a,b]$ then the following inequalities hold almost everywhere*

$$X\left(\frac{2ab}{a+b},\cdot\right) \le \frac{ab}{b-a}\int_a^b \frac{X(t,\cdot)}{t^2}dt \le \frac{X(a,\cdot) + X(b,\cdot)}{2}. \tag{2.2}$$

Let us consider the two-dimensional interval $\Delta = [a,b] \times [c,d]$ in $(0,\infty) \times (0,\infty)$ with $a < b$ and $c < d$.

**Definition 2.5[12].** *A stochastic process $X: \Delta \times \Omega \to \mathbb{R}$ is said to be a harmonically convex on $\Delta$, if the following inequality holds almost everywhere*

$$X\left(\left(\frac{t_1t_2}{\lambda t_1+(1-\lambda)t_2},\frac{s_1s_2}{\lambda s_1+(1-\lambda)s_2}\right),\cdot\right)\leq \lambda X\big((t_2,s_2),\cdot\big)+(1-\lambda)X\big((t_1,s_1),\cdot\big)$$

for all $(t_1,s_1),(t_2,s_2)\in\Delta$ and $\lambda\in[0,1]$. If the above inequality is reversed, then $X$ is said to be a harmonically concave on $\Delta$.

**Definition 2.6[12].** *A stochastic process $X:\Delta\times\Omega\to\mathbb{R}$ is said to be a harmonically convex on the co-ordinates on $\Delta\times\Omega$ if the partial mappings $X_s:[a,b]\times\Omega\to\mathbb{R}, X_s(u,\cdot):=X\big((u,s),\cdot\big)$ and $X_t:[c,d]\times\Omega\to\mathbb{R}, X_t(v,\cdot):=X((t,v),\cdot)$ defined for all $t\in[a,b]$ and $s\in[c,d]$ are harmonically convex almost everywhere.*

**Theorem 2.1[12].** *Suppose that $X:\Delta\times\Omega\to\mathbb{R}$ is harmonically convex on the coordinates on $\Delta$. Then the following inequalities hold almost everywhere:*

$$X\left(\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right),\cdot\right)$$

$$\leq\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}X\left(\left(t,\frac{2cd}{c+d}\right),\cdot\right)dt+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}X\left(\left(\frac{2ab}{a+b},s\right),\cdot\right)ds\right]$$

$$\leq\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{1}{(ts)^2}X((t,s),.)dtds$$

$$\leq\frac{1}{4}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}\Big(X\big((t,c),\cdot\big)+X\big((t,d),\cdot\big)\Big)dt+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}\Big(X\big((a,s),\cdot\big)+X\big((b,s),\cdot\big)\Big)ds\right]$$

$$\leq\frac{X\big((a,c),\cdot\big)+X\big((b,c),\cdot\big)+X\big((a,d),\cdot\big)+X\big((b,d),\cdot\big)}{4}\qquad(2.3)$$

For $n\geq 2$, let $u_i,v_i,(i=1,2,\dots,n)$ be real numbers such that $u_i<v_i$ for $i=1,2,\dots,n$, and the n-dimensional interval $\Delta^n=\prod_{i=1}^n[u_i,v_i]\subseteq[0,\infty)^n$.

In the following we give definition of convexity for stochastic processes on n-coordinates:

**Definition 2.7[13].** *A stochastic process $X:\Delta^n\times\Omega\to\mathbb{R}$ is called convex on n-coordinates if the stochastic processes*

$$X_{t_n}^i(t,\cdot):=X\big((t_1,\dots,t_{i-1},t,t_{i+1},\dots,t_n),\cdot\big)$$

*are convex on $[u_i,v_i]$ almost everywhere for $i=1,2,\dots,n$.*

**Definition 2.8[13].** *A stochastic process $X:\Delta^n\times\Omega\to\mathbb{R}$ is said to be convex on $\Delta^n$ if the following inequality holds almost everywhere*

$$X\big((\lambda\boldsymbol{t}+(1-\lambda)\boldsymbol{s}),\cdot\big)\leq\lambda X(\boldsymbol{t},\cdot)+(1-\lambda)X(\boldsymbol{s},\cdot)$$

*for all $\boldsymbol{t}=(t_1,t_2,\dots,t_n),\boldsymbol{s}=(s_1,s_2,\dots,s_n)\in\Delta^n$ and $\lambda\in[0,1]$. If the above inequality is reversed then $X$ is said to be concave on $\Delta^n$.*

## 3. MAIN RESULTS

The main goal of this section is to present Hermite-Hadamard type inequalities for harmonically convex stochastic processes on n-coordinates.

In the following we give definition of harmonically convexity for stochastic processes on n-coordinates:

**Definition 3.1.** *A stochastic process $X\colon \Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ is called convex on n-coordinates if the stochastic processes*

$$X_{t_n}^i(t,\cdot) \coloneqq X\big((t_1,\dots,t_{i-1},t,t_{i+1},\dots,t_n),\cdot\big)$$

*are* harmonically *convex on $[u_i,v_i]$ almost everywhere for $i=1,2,\dots,n$.*

**Definition 3.2.** *A stochastic process $X\colon \Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ is said to be harmonically convex on $\Delta^n$ if the following inequality holds almost everywhere*

$$X\left(\left(\frac{\boldsymbol{ts}}{\lambda\boldsymbol{t}+(1-\lambda)\boldsymbol{s}}\right),\cdot\right) \le \lambda X(\boldsymbol{s},\cdot)+(1-\lambda)X(\boldsymbol{t},\cdot) \tag{3.1}$$

*for all $\boldsymbol{t}=(t_1,t_2,\dots,t_n),\boldsymbol{s}=(s_1,s_2,\dots,s_n)\in\Delta^n$ and $\lambda\in[0,1]$. If the above inequality is reversed then X is said to be harmonically concave on $\Delta^n$.*

**Proposition 3.1.** *Let be $X\colon \Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ is a stochastic process and $X_{t_n}^i(t,\cdot) \coloneqq X\big((t_1,\dots,t_{i-1},t,t_{i+1},\dots,t_n),\cdot\big)$, for some $i=1,2,\dots,n$.*

*(i) If $X_{t_n}^i$ is convex and nondecreasing stochastic process for $i=1,2,\dots,n$, then X is harmonically convex on n-coordinates.*

*(ii) If $X_{t_n}^i$ is harmonically convex and nonincreasing stochastic process for $i=1,2,\dots,n$, then X is convex on n-coordinates.*

**Lemma 3.1.** *Every harmonically convex stochastic process $X\colon \Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ is* harmonically *convex on n-coordinates almost everywhere but converse is not true.*

**Proof.** Let $X\colon \Delta^n \times \Omega \to \mathbb{R}$ be harmonically convex *on* $\Delta^n$. Assume that $X_{t_n}^i\colon [u_i,v_i]\times\Omega \to \mathbb{R}$, defined by

$$X_{t_n}^i(t,\cdot) \coloneqq X\big((t_1,\dots,t_{i-1},t,t_{i+1},\dots,t_n),\cdot\big), \qquad t\in[u_i,v_i].$$

Now for $t,s\in[u_i,v_i]$ and $\lambda\in[0,1]$ almost everywhere

$$X_{t_n}^i\left(\left(\frac{ts}{\lambda t+(1-\lambda)s}\right),\cdot\right) = X\left(\left(t_1,\dots,t_{i-1},\frac{ts}{\lambda t+(1-\lambda)s},t_{i+1},\dots,t_n\right),\cdot\right)$$

$$= X\left(\left(\begin{matrix}\dfrac{t_1^2}{\lambda t_1+(1-\lambda)t_1},\dots,\dfrac{t_{i-1}^2}{\lambda t_{i-1}+(1-\lambda)t_{i-1}},\dfrac{ts}{\lambda t+(1-\lambda)s},\\ \dfrac{t_{i+1}^2}{\lambda t_{i+1}+(1-\lambda)t_{i+1}},\dots,\dfrac{t_n^2}{\lambda t_n+(1-\lambda)t_n}\end{matrix}\right),\cdot\right)$$

$$\leq \lambda X\big((t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n),\cdot\big) + (1-\lambda)X\big((t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n),\cdot\big)$$

$$= \lambda X_{t_n}^{i}(s,\cdot) + (1-\lambda)X_{t_n}^{i}(t,\cdot)$$

which implies $X_{t_n}^{i}$ is harmonically convex on $[u_i, v_i]$, that is, $X$ is harmonically convex on n-coordinates.

Now, consider $X: [3,6] \times [4,6] \times [5,6] \times \Omega \to [0,\infty)$ given $X\big((t_1, t_2, t_3),\cdot\big) = (t_1 - 3)(t_1 - 4)(t_3 - 5)$. It is obvious that $X$ harmonically convex on 3-coordinates by Proposition 3.1, but is not harmonically convex on $[3,6] \times [4,6] \times [5,6]$. Indeed $(3,6,6), (4,6,5) \in [3,6] \times [4,6] \times [5,6]$ and $\lambda \in (0,1)$ we have

$$X\left(\left(\frac{3.4}{3.\lambda + (1-\lambda).4}, \frac{6.6}{6.\lambda + (1-\lambda).6}, \frac{6.5}{6.\lambda + (1-\lambda).5}\right),\cdot\right)$$

$$= X\left(\left(\frac{12}{4-\lambda}, 6, \frac{30}{5+\lambda}\right),\cdot\right) = \left(\frac{12}{4-\lambda} - 3\right)(6-4)\left(\frac{30}{5+\lambda} - 5\right)$$

$$= \left(\frac{3\lambda}{4-\lambda}\right)2\left(\frac{5-\lambda}{5+\lambda}\right) = \frac{6\lambda(5-\lambda)}{(4-\lambda)(5+\lambda)} > 0;$$

and

$$\lambda X\big((3,6,6),\cdot\big) + (1-\lambda)X\big((4,6,5),\cdot\big)$$

$$= \lambda(3-3)(6-4)(6-5) + (1-\lambda)(4-3)(6-4)(5-5) = 0.$$

Thus for all $\lambda \in (0,1)$, we have

$$X\left(\left(\frac{3.4}{3.\lambda + (1-\lambda).4}, \frac{6.6}{6.\lambda + (1-\lambda).6}, \frac{6.5}{6.\lambda + (1-\lambda).5}\right),\cdot\right)$$

$$> \lambda X\big((3,6,6),\cdot\big) + (1-\lambda)X\big((4,6,5),\cdot\big)$$

which shows that $X$ is not harmonically convex on $[3,6] \times [4,6] \times [5,6]$.

**Theorem 3.1.** *If $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ is harmonically convex stochastic process on n-coordinates, then $X_{t_n}^{k}: [u_k, v_k] \times \Omega \to \mathbb{R}$ is convex on $[u_k, v_k]$ for each $k = 1,2, \ldots, n$. From Hermite-Hadamard inequality, we have almost everywhere*

$$\sum_{k=1}^{n-1} X\left(\left(t_1, \ldots, t_{k-1}, \frac{2u_k v_k}{v_k + u_k}, \frac{2u_{k+1}v_{k+1}}{v_{k+1} + u_{k+1}}, \ldots, t_n\right),\cdot\right)$$

$$\leq \sum_{k=1}^{n-1} \frac{u_k v_k}{v_k - u_k} \int_{u_k}^{v_k} \frac{X_{t_n}^{k}\left(\frac{2u_{k+1}v_{k+1}}{u_{k+1} + v_{k+1}},\cdot\right)}{t_k^2} dt_k$$

$$\leq \sum_{k=1}^{n-1} \frac{u_k v_k u_{k+1} v_{k+1}}{(v_k - u_k)(v_{k+1} - u_{k+1})} \int_{u_k}^{v_k} \int_{u_{k+1}}^{v_{k+1}} \frac{X_{t_n}^{k+1}(t_{k+1},\cdot)}{(t_k t_{k+1})^2} dt_{k+1} dt_k$$

$$\leq \sum_{k=1}^{n-1} \frac{u_k v_k}{v_k - u_k} \int_{u_k}^{v_k} \frac{X_{t_n}^k(u_{k+1},\cdot) + X_{t_n}^k(v_{k+1},\cdot)}{2t_k^2} dt_k$$

$$\leq \frac{1}{4} \sum_{k=1}^{n-1} \begin{bmatrix} X\big((t_1, \dots, t_{k-1}, u_k, u_{k+1}, \dots, t_n),\cdot\big) + X\big((t_1, \dots, t_{k-1}, v_k, u_{k+1}, \dots, t_n),\cdot\big) \\ + X\big((t_1, \dots, t_{k-1}, u_k, v_{k+1}, \dots, t_n),\cdot\big) + X\big((t_1, \dots, t_{k-1}, v_k, v_{k+1}, \dots, t_n),\cdot\big) \end{bmatrix}. \quad (3.2)$$

**Proof.** Since $X: \Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ is harmonically convex stochastic on n-coordinates, there for $X_{t_n}^i: [u_i, v_i] \times \Omega \to \mathbb{R}$ is harmonically convex stochastic process on $[u_i, u_i]$ for each $i = 1,2, \dots, n$, we have on $[u_{i+1}, v_{i+1}]$ almost everywhere

$$X_{t_n}^{i+1}\left(\frac{2u_{i+1}v_{i+1}}{u_{i+1} + v_{i+1}},\cdot\right) \leq \frac{u_{i+1}v_{i+1}}{v_{i+1} - u_{i+1}} \int_{u_{i+1}}^{v_{i+1}} \frac{X_{t_n}^{i+1}(t_{i+1},\cdot)}{t_{i+1}^2} dt_{i+1} \leq \frac{X_{t_n}^{i+1}(u_{i+1},\cdot) + X_{t_n}^{i+1}(v_{i+1},\cdot)}{2}.$$

All of sides of the above inequalities by integrating over $[u_i, u_i]$

$$\frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}\left(\frac{2u_{i+1}v_{i+1}}{u_{i+1} + v_{i+1}},\cdot\right)}{t_i^2} dt_i$$

$$\leq \frac{u_i v_i u_{i+1} v_{i+1}}{(v_i - u_i)(v_{i+1} - u_{i+1})} \int_{u_i}^{v_i} \int_{u_{i+1}}^{v_{i+1}} \frac{X_{t_n}^{i+1}(t_{i+1},\cdot)}{t_{i+1}^2} dt_{i+1} dt_i$$

$$\leq \frac{u_i v_i}{2(v_i - u_i)} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1},\cdot) + X_{t_n}^{i+1}(v_{i+1},\cdot)}{t_i^2} dt_i. \quad (3.3)$$

Again applying the Hermite-Hdamard inequality

$$X\left(\left(t_1, \dots, \frac{2u_i v_i}{v_i + u_i}, \frac{2u_{i+1}v_{i+1}}{u_{i+1} + v_{i+1}}, \dots, t_n\right),\cdot\right) \leq \frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}\left(\frac{2u_{i+1}v_{i+1}}{u_{i+1} + v_{i+1}},\cdot\right)}{t_i^2} dt_i \quad (3.4)$$

for each $i \in \{1,2, \dots, n-1\}$ and also

$$\frac{u_i v_i}{2(v_i - u_i)} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1},\cdot) + X_{t_n}^{i+1}(v_{i+1},\cdot)}{t_i^2} dt_i$$

$$= \frac{1}{2}\left[\frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1},\cdot)}{t_i^2} dt_i + \frac{u_i v_i}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(v_{i+1},\cdot)}{t_i^2} dt_i\right]$$

$$\leq \frac{1}{2} \begin{bmatrix} \dfrac{X\big((t_1, \dots, t_{i-1}, u_i, u_{i+1}, \dots, t_n),\cdot\big) + X\big((t_1, \dots, t_{i-1}, v_i, u_{i+1}, \dots, t_n),\cdot\big)}{2} \\ + \dfrac{X\big((t_1, \dots, t_{i-1}, u_i, v_{i+1}, \dots, t_n),\cdot\big) + X\big((t_1, \dots, t_{i-1}, v_i, v_{i+1}, \dots, t_n),\cdot\big)}{2} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} X\big((t_1, \dots, t_{i-1}, u_i, u_{i+1}, \dots, t_n),\cdot\big) + X\big((t_1, \dots, t_{i-1}, v_i, u_{i+1}, \dots, t_n),\cdot\big) \\ + X\big((t_1, \dots, t_{i-1}, u_i, v_{i+1}, \dots, t_n),\cdot\big) + X\big((t_1, \dots, t_{i-1}, v_i, v_{i+1}, \dots, t_n),\cdot\big) \end{bmatrix} \quad (3.5)$$

for each $i \in \{1,2, \dots, n-1\}$. Using the inequalities (3.4) and (3.5) in (3.3) and taking summation from 1 to $n-1$, we have (3.2).

**Corollary 3.1.** *Let $X:\Delta^2 \subset \mathbb{R}_+^2 \times \Omega \to \mathbb{R}$ be harmonically convex stochastic process on 2-coordinates. Then (2.3) is valid.*

**Theorem 3.2.** *Let $X:\Delta^n \subset \mathbb{R}_+^n \times \Omega \to \mathbb{R}$ be harmonically convex stochastic process on n-coordinates. Then we have almost everywhere*

$$X\left(\left(\frac{2u_1v_1}{v_1+u_1},\dots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{v_n+u_n}\right),\cdot\right)$$

$$\leq \left(\prod_{i=1}^{n}\frac{u_iv_i}{v_i-u_i}\right)\int_{u_1}^{v_1}\dots\int_{u_n}^{v_n}\frac{X_{t_n}^n(t_n,\cdot)}{(\prod_{i=1}^{n}t_i)^2}dt_n\dots dt_1 \leq \frac{1}{2^n}\sum_{\delta\in l_i(n)}X(\delta u+(1-\delta)v,\cdot), \quad (3.6)$$

where $l_i(n) \coloneqq \{\delta \in \mathbb{N}_0^n : \delta \leq 1, |\delta| = n+1-i,\ i=1,\dots,n+1\}$, ; $|\delta| \coloneqq \delta_1+\dots+\delta_n \in \mathbb{N}$; $\delta u := (\delta_1u_1,\dots,\delta_nu_n) \in \mathbb{N}_0^n$ for $u,v \in \Delta^n$.

**Proof.** By applying Hermite-Hadamard's inequality for harmonically convex stochastic process $X_{t_n}^n$ on interval $[u_n, v_n]$ we have almost everywhere

$$X_{t_n}^n\left(\frac{2u_nv_n}{u_n+v_n},\cdot\right) \leq \frac{u_nv_n}{v_n-u_n}\int_{u_n}^{v_n}\frac{X_{t_n}^n(t_n,\cdot)}{t_n^2}dt_n \leq \frac{X_{t_n}^n(u_n,\cdot)+X_{t_n}^n(v_n,\cdot)}{2}.$$

By integrating on $[u_{n-1}, v_{n-1}]$, we get

$$\frac{u_{n-1}v_{n-1}}{v_{n-1}-u_{n-1}}\int_{u_{n-1}}^{v_{n-1}}\frac{X_{t_n}^n\left(\frac{2u_nv_n}{u_n+v_n},\cdot\right)}{t_{n-1}^2}dt_{n-1}$$

$$\leq \frac{u_{n-1}v_{n-1}u_nv_n}{(v_{n-1}-u_{n-1})(v_n-u_n)}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n}\frac{X_{t_n}^n(t_n,\cdot)}{(t_{n-1}t_n)^2}dt_{n-1}dt_n$$

$$\leq \frac{u_{n-1}v_{n-1}}{v_{n-1}-u_{n-1}}\int_{u_{n-1}}^{v_{n-1}}\frac{X_{t_n}^n(u_n,\cdot)+X_{t_n}^n(v_n,\cdot)}{2t_{n-1}^2}dt_{n-1}. \quad (3.7)$$

From (3.4), (3.5), respectively

$$X\left(\left(t_1,\dots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{u_n+v_n}\right),\cdot\right) \leq \frac{u_{n-1}v_{n-1}}{v_{n-1}-u_{n-1}}\int_{u_{n-1}}^{v_{n-1}}\frac{X_{t_n}^n\left(\frac{2u_nv_n}{u_n+v_n},\cdot\right)}{t_{n-1}^2}dt_{n-1}, \quad (3.8)$$

$$\frac{u_{n-1}v_{n-1}}{2(v_{n-1}-u_{n-1})}\int_{u_{n-1}}^{v_{n-1}}\frac{X_{t_n}^n(u_n,\cdot)+X_{t_n}^n(v_n,\cdot)}{t_{n-1}^2}dt_{n-1}$$

$$\leq \frac{1}{2^2}\begin{bmatrix} X\big((t_1,\dots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},v_n),\cdot\big)\end{bmatrix}. \quad (3.9)$$

From (3.7)-(3.9)

$$X\left(\left(t_1,\dots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{u_n+v_n}\right),\cdot\right)$$

$$\leq \frac{u_{n-1}v_{n-1}u_nv_n}{(v_{n-1}-u_{n-1})(v_n-u_n)}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n}\frac{X_{t_n}^n(t_n,\cdot)}{(t_{n-1}t_n)^2}dt_{n-1}dt_n$$

$$\leq \frac{1}{2^2}\begin{bmatrix} X\big((t_1,\dots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},v_n),\cdot\big)\end{bmatrix}. \tag{3.10}$$

Thus, integrating on $[u_{n-2},v_{n-2}]$, we have

$$\frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{X\left(\left(t_1,\dots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{u_n+v_n}\right),\cdot\right)}{t_{n-2}^2}dt_{n-2}$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{u_iv_i}{v_i-u_i}\right)\int_{u_{n-2}}^{v_{n-2}}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n}\frac{X_{t_n}^n(t_n,\cdot)}{(\prod_{i=n-2}^{n}t_i)^2}dt_{n-2}dt_{n-1}dt_n$$

$$\leq \frac{1}{2^2}\frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{1}{t_{n-2}^2}\begin{bmatrix} X\big((t_1,\dots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},v_n),\cdot\big)\end{bmatrix}dt_{n-2}. \tag{3.11}$$

From (3.4), (3.5), respectively

$$X\left(\left(t_1,\dots,\frac{2u_{n-2}v_{n-2}}{v_{n-2}+u_{n-2}},\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{u_n+v_n}\right),\cdot\right)$$

$$\leq \frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{X\left(\left(t_1,\dots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{u_n+v_n}\right),\cdot\right)}{t_{n-2}^2}dt_{n-2}\,, \tag{3.12}$$

$$\leq \frac{1}{2^2}\frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{1}{t_{n-2}^2}\begin{bmatrix} X\big((t_1,\dots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},v_n),\cdot\big)\end{bmatrix}dt_{n-2}$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X\big((t_1,\dots,u_{n-2},u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-2},u_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-2},v_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-2},v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-2},u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-2},u_{n-1},v_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-2},v_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-2},v_{n-1},v_n),\cdot\big)\end{bmatrix}. \tag{3.13}$$

From (3.11)-(3.13)

$$\frac{u_{n-2}v_{n-2}}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{X\left(\left(t_1,\dots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{u_n+v_n}\right),\cdot\right)}{t_{n-2}^2}dt_{n-2}$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{u_iv_i}{v_i-u_i}\right)\int_{u_{n-2}}^{v_{n-2}}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n}\frac{X_{t_n}^n(t_n,\cdot)}{(\prod_{i=n-2}^{n}t_i)^2}dt_{n-2}dt_{n-1}dt_n$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X\big((t_1,\dots,u_{n-2},u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,u_{n-2},v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-2},u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,u_{n-2},v_{n-1},v_n),\cdot\big) \\ X\big((t_1,\dots,v_{n-2},u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-2},v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,v_{n-2},u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-2},v_{n-1},v_n),\cdot\big) \end{bmatrix}. \tag{3.14}$$

Using inductive method for $n = k-1$, we obtain

$$X\left(\left(\frac{2u_1v_1}{v_1+u_1},\dots,\frac{2u_{k-2}v_{k-2}}{v_{k-2}+u_{k-2}},\frac{2u_{k-1}v_{k-1}}{v_{k-1}+u_{k-1}}\right),\cdot\right)$$

$$\leq \left(\prod_{i=1}^{k-1}\frac{u_iv_i}{v_i-u_i}\right)\int_{u_1}^{v_1}\dots\int_{u_{k-1}}^{v_{k-1}}\frac{X_{t_{k-1}}^{k-1}(t_{k-1},\cdot)}{\left(\prod_{i=1}^{k-1}t_i\right)^2}dt_n\dots dt_1 \leq \frac{1}{2^{k-1}}\sum_{\delta\in l_i(k-1)}X(\delta u+(1-\delta)v,\cdot)$$

where $l_i(k-1) \coloneqq \{\delta\in\mathbb{N}_0^n: \delta\leq 1, |\delta| = k-i;\ i=1,\dots,k, k\in\mathbb{N}\}$, $|\delta| \coloneqq \delta_1+\dots+\delta_n \in \mathbb{N}$; $\delta u := (\delta_1u_1,\dots,\delta_nu_n)\in\mathbb{N}_0^n$ for $u,v\in\Delta^n$. Consequently, for $n=k$, we get (3.6).

**Example 1.** *Let $X\colon \Delta^3\times\Omega\to\mathbb{R}$ be a harmonically convex stochastic process and be integrated in mean-square on $\Delta^3$. Then almost everywhere*

$$X\left(\left(\frac{2u_1v_1}{v_1+u_1},\frac{2u_2v_2}{v_2+u_2},\frac{2u_3v_3}{v_3+u_3}\right),\cdot\right)$$

$$\leq \frac{u_1v_1u_2v_2u_3v_3}{(v_1-u_1)(v_2-u_2)(v_3-u_3)}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\int_{u_3}^{v_3}\frac{X\big((t_1,t_2,t_3),\cdot\big)}{t_1^2t_2^2t_3^2}dt_3dt_2dt_1$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X\big((u_1,u_2,u_3),\cdot\big)+X\big((v_1,u_2,u_3),\cdot\big) \\ +X\big((u_1,v_2,u_3),\cdot\big)+X\big((v_1,v_2,u_3),\cdot\big) \\ +X\big((u_1,u_2,v_3),\cdot\big)+X\big((v_1,u_2,v_3),\cdot\big) \\ +X\big((u_1,v_2,v_3),\cdot\big)+X\big((v_1,v_2,v_3),\cdot\big) \end{bmatrix}.$$

**Proof.** According to Theorem 2 for n = 3, we get $X_{t_3}^3(t_3,\cdot) \coloneqq X\big((t_1,t_2,t_3),\cdot\big)$ and $l_i(3) \coloneqq \{\delta\in\mathbb{N}_0^3: \delta\leq 1, |\delta| = 4-i\}, i = 1,2,3,4$. Then

$$l_1(3) = \{(1,1,1)\};\ l_2(3) = \{(0,1,1)\,,(1,0,1),(1,1,0)\},$$

$$l_3(3) = \{(0,0,1)\,,(0,1,0),(1,0,0),\};l_4(3) = \{(0,0,0)\},$$

and for $u = (u_1,u_2,u_3)$, $v = (v_1,v_2,v_3)\in\Delta^3$

$$\sum_{\delta\in l_1(3)}X(\delta u+(1-\delta)v,\cdot)$$

$$= X\big((1,1,1)(u_1,u_2,u_3)+[(1,1,1)-(1,1,1)](v_1,v_2,v_3),\cdot\big) = X\big((u_1,u_2,u_3),\cdot\big);$$

$$\sum_{\delta\in l_2(3)}X(\delta u+(1-\delta)v,\cdot) = X\big((0,1,1)(u_1,u_2,u_3)+[(1,1,1)-(0,1,1)](v_1,v_2,v_3),\cdot\big)$$

$$+X\big((1,0,1)(u_1,u_2,u_3)+[(1,1,1)-(1,0,1)](v_1,v_2,v_3),\cdot\big)$$

$$+X\big((1,1,0)(u_1,u_2,u_3)+[(1,1,1)-(1,1,0)](v_1,v_2,v_3),\cdot\big)$$

$$=X\big((u_1,v_2,v_3),\cdot\big)+X\big((u_1,v_2,u_3),\cdot\big)+X\big((u_1,u_2,v_3),\cdot\big);$$

So,

$$\sum_{\delta\in l_3(3)} X(\delta u+(1-\delta)v,\cdot)=X\big((v_1,v_2,u_3),\cdot\big)+X\big((v_1,u_2,v_3),\cdot\big)+X\big((u_1,v_2,v_3),\cdot\big);$$

$$\sum_{\delta\in l_4(3)} X(\delta u+(1-\delta)v,\cdot)=X\big((v_1,v_2,v_3),\cdot\big).$$

Thus

$$\sum_{\delta\in l_i(3)} X(\delta u+(1-\delta)v,\cdot)=X\big((u_1,u_2,u_3),\cdot\big)$$

$$+X\big((u_1,v_2,v_3),\cdot\big)+X\big((u_1,v_2,u_3),\cdot\big)+X\big((u_1,u_2,v_3),\cdot\big)$$

$$+X\big((v_1,v_2,u_3),\cdot\big)+X\big((v_1,u_2,v_3),\cdot\big)+X\big((u_1,v_2,v_3),\cdot\big)+X\big((v_1,v_2,v_3),\cdot\big).$$

Using all of the above equalities in (3.6), we obtain the desired result in this example.

## 4. CONCLUSION

In this paper, we obtain some new Hermite-Hadamard type inequalities for harmonically convex stochastic processes on n-coordinates and obtain. As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes.

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